

Introduction¹ to Modified Gravity: from the Cosmic Speedup Problem to Quantum Gravity Phenomenology

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Abstract

These notes represent a summary of the introductory part of a course on modified gravity delivered at several Spanish Universities (Granada, Valencia, and Valladolid), at the University of Wisconsin-Milwaukee (WI, USA), and at the Karl-Franzens Universität (Graz, Austria) during the period 2008-2011. We begin with a discussion of the classical Newtonian framework and how special relativity boosted the interest on new theories of gravity. Then we focus on Nordström's scalar theories of gravity and their influence on Einstein's theory of general relativity. We comment on the meaning of the Einstein equivalence principle and its implications for the construction of alternative theories of gravity. We present the cosmic speedup problem and how $f(R)$ theories can be constrained attending to their weak-field behavior. We conclude by showing that Palatini $f(R)$ and $f(R, Q)$ theories can be used to address different aspects of quantum gravity phenomenology and singularity problems.

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1.1 Introduction

The reasons and motivations that lead to the consideration of alternatives to General Relativity are manifold and have changed over the years. Some theories are motivated by theoretical reasons while others are more phenomenological. One can thus find theories aimed at unifying different interactions, such as Kaluza-Klein theory (5-dimensional spacetime as a possible framework to unify gravitation and electromagnetism) or the very famous string theory (which should provide a unified explanation for *everything*, i.e., from particles to interactions); others appeared as spin-offs of string theory and are now seen as independent frameworks for testing some of its phenomenology, such is the case of the string-inspired “brane worlds” (which confine the standard model of elementary particles to a 4-dimensional brane within a larger bulk accessible to gravitational interactions); we also find modifications of GR needed to allow for its perturbative renormalization, or modifications aimed at avoiding the big bang singularity, effective actions related with non-perturbative quantization schemes, etcetera. All them are motivated by theoretical problems.

On the other hand, we find theories motivated by the need to find alternative explanations for the current cosmological model and astrophysical observations,

which depict a Universe filled with some kind of *aether* or *dark energy* representing the main part of the energy budget of the Universe, followed by huge amounts of *unseen* matter which seems necessary to explain the anomalous rotation curves of galaxies, gravitational lensing, and the formation of structure via gravitational instability.

One of the goals of this chapter is to provide the reader with elementary concepts and tools that will allow him/her better understand different alternatives to GR recently considered in the literature in relation with the cosmic speedup problem and the phenomenology of quantum gravity during the very early universe. Since such theories are aimed at explaining certain observational facts, they must be able to account for the new effects they have been proposed for but also must be compatible with other observational and experimental constraints coming from other scenarios. The process of building and testing these theories is, in our opinion, a very productive theoretical exercise, since it allows us to give some freedom to our imagination but at the same time forces us to keep our feet on the ground.

Though there are no limits to imagination, experiments and observations should be used as a guide to build and put limitations on sensible theories. In fact, a careful theoretical interpretation of experiments can be an excellent guide to constrain the family of viable theories. In this sense, we believe it is extremely important to clearly understand the implications of the Einstein equivalence principle (EEP). We hope these notes manage to convey the idea that theorists should have a deep knowledge and clear understanding of the experiments related with gravitation.

We believe that $f(R)$ theories of gravity are a nice toy model to study a possible gravitational alternative to the *dark energy problem*. Their dynamics is relatively simple and they can be put into correspondence with scalar-tensor theories of gravity, which appear in many different contexts in gravitational physics, from extended inflation and extended quintessence models to Kaluza-Klein and string theory. On the other hand, $f(R)$ theories, in the Palatini version, also seem to have some relation with non-perturbative approaches to quantum gravity. Though such approaches have only been applied with certain confidence in highly symmetric scenarios (isotropic and anisotropic, homogeneous cosmologies) they indicate that the Big Bang singularity can be avoided quite generally without the need for extra degrees of freedom. Palatini $f(R)$ theories can also be designed to remove that singularity and reproduce the dynamical equations derived from isotropic models of Loop Quantum Cosmology. Extended Lagrangians of the form $f(R, Q)$, being Q the squared Ricci tensor, exhibit even richer phenomenology than Palatini $f(R)$ models. These are very interesting and promising aspects of these theories of gravity that are receiving increasing attention in the recent literature and that will be treated in detail in these lectures.

We begin with Newton's theory, the discovery of special relativity, and Nordström's scalar theories as a way to motivate the idea of gravitation as a curved space phenomenon. Once the foundations of gravitation have been settled, we shift our attention to the predictions of particular theories, paying special at-

tention to $f(R)$ theories and some extensions of that family of theories. We show how the solar system dynamics can be used to reconstruct the form of the gravity Lagrangian and how modified gravity can be useful in modeling certain aspects of quantum gravity phenomenology.

1.2 From Newtonian physics to Einstein's gravity.

In his *Principia Mathematica* (1687) Newton introduced the fundamental three laws of classical mechanics:

- If no net force acts on a particle, then it is possible to select a set of reference frames (inertial frames), observed from which the particle moves without any change in velocity. This is the so called *Principle of Relativity* (PoR).
- From an inertial frame, the net force on a particle of mass m is $\vec{F} = m\vec{a}$.
- Whenever a particle A exerts a force on another particle B, B simultaneously exerts a force on A with the same magnitude in the opposite direction.

Using Newton's laws one could explain all kinds of motion. When a nonzero force acts on a body, it accelerates at a rate that depends on its *inertial* mass m_i . A given force will thus lead to different accelerations depending on the inertial mass of the body. In his *Principia*, Newton also found an explanation to Kepler's empirical laws of planetary motion: between any two bodies separated by a distance d , there exists a force called *gravity* given by $F_g = G \frac{m_1 m_2}{d^2}$. Here G is a constant, and m_1, m_2 represent the *gravitational* masses of those bodies. When one studies experimentally Newton's theory of gravity quickly realizes that there is a deep relation between the inertial and the gravitational mass of a body. It turns out that the acceleration a experienced by any two bodies on the surface of the Earth looks the same irrespective of the mass of those bodies. This suggests that inertial and gravitational mass have the same numerical values, $m_i = m_g$ (in general, they are proportional, being the proportionality constant the same for all bodies). This observation is known as *Newton's equivalence principle* or weak equivalence principle.

From Newton's laws it follows that Newtonian physics is based on the idea of *absolute space*, a background structure with respect to which accelerations can be effectively measured. However, the PoR implies that, unlike accelerations, absolute positions and velocities are not directly observable. This conclusion was challenged by some results published in 1865 by J.C. Maxwell. In Maxwell's work, the equations of the electric and magnetic field were improved by the addition of a new term (Maxwell's displacement current). The new equations predicted the existence of electromagnetic waves. The explicit appearance in those equations of a speed c was interpreted as the existence of a privileged

reference frame, that of the *luminiferous aether*¹. According to this, it could be possible to measure absolute velocities (at least with respect to the aether²). This idea motivated the experiment carried out by Michelson and Morley in 1887 to measure the relative velocity of the Earth in its orbit around the sun with respect to the aether³. Despite the experimental limitations of the epoch, their experiment had enough precision to confirm that the speed of light is independent of the direction of the light ray and the position of the Earth in its orbit.

Motivated by this intriguing phenomenon, in 1892 Lorentz proposed that moving bodies contract in the direction of motion according to a specific set of transformations. In 1905 Einstein presented its celebrated theory of special relativity and derived the Lorentz transformations using the PoR and the observed constancy of the speed of light without assuming the presence of an aether. Therefore, though the principle of relative motion had been put into question by electromagnetism, it was salvaged by Einstein's reinterpretation⁴. As of that moment, it was understood that any good physical theory should be adapted to the new PoR. Fortunately, Minkowski (1907) realized that Lorentz transformations could be nicely interpreted in a four dimensional space-time (he thus invented the notion of spacetime as opposed to the well-known spatial geometry of the time). In this manner, a Lorentz-invariant theory should be constructed using geometrical invariants such as scalars and four-vectors, which represents a geometrical formulation of the PoR.

1.2.1 A relativistic theory of gravity: Nordström's theory.

The acceptance of the new PoR led to the development of relativistic theories of gravity in which the gravitational field was represented by different types of fields, such as scalars (in analogy with Newtonian mechanics) or vectors (in analogy with Maxwell's electrodynamics). A natural proposal⁵ in this sense consists on replacing the Newtonian equations by the following relativistic versions [Norton, 1992]

$$\nabla^2 \phi = 4\pi G\rho \rightarrow \square \phi = 4\pi G\rho \quad (1.1)$$

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi \rightarrow \frac{du_\mu}{d\tau} = -\partial_\mu \phi \quad (1.2)$$

¹The aether was supposed to have very special properties, such as a very high elasticity, and to exhibit no friction to the motion of bodies through it.

²The aether was assumed to be at rest because otherwise the light from distant stars would suffer distortions in their propagation due to local motions of this fluid.

³Note that the speed of sound is relative to the wind. Analogously, it was thought that the speed of light should be measured with respect to the aether. Due to the motion of the Earth, that speed should depend on the position of the Earth and the direction of the light ray. The interferometer was built on a rotating surface such that the full experiment could be rotated to observe periodic variations of the interference pattern.

⁴It is worth noting that Einstein's results did not rule out the aether, but they implied that its presence was irrelevant for the discussion of experiments.

⁵Another very natural proposal would be a relativistic theory of gravity inspired by Maxwell's electrodynamics, being $F_\mu \equiv mdu_\mu/d\tau = kG_{\mu\nu}u^\nu$ with $G_{\mu\nu} = -G_{\nu\mu}$. Such a proposal immediately implies that $F_\mu u^\mu = 0$ and is compatible with the constancy of c^2 .

This proposal, however, is unsatisfactory. From the assumed constancy of the speed of light, $\eta_{\mu\nu}u^\mu u^\nu = -c^2$, one finds that $u_\mu \frac{du^\mu}{d\tau} = 0$, which implies the unnatural restriction $u^\mu \partial_\mu \phi = \frac{d\phi}{d\tau} = 0$, i.e., the gravitational field should be constant along any observer's world line.

To overcome this drawback, Nordström proposed that the mass of a body in a gravitational field could vary with the gravitational potential [Nordström, 1912]. Nordström proposed a relativistic scalar theory of gravity in which the matter evolution equation (1.2) was modified to make it compatible with the constancy of the speed of light

$$F_\mu \equiv \frac{d(mu_\mu)}{d\tau} = -m\partial_\mu \phi \leftrightarrow m \frac{du_\mu}{d\tau} + u_\mu \frac{dm}{d\tau} = -m\partial_\mu \phi. \quad (1.3)$$

This equation implies that in a gravitational field m changes as $md\phi/d\tau = c^2 dm/d\tau$, which leads to $m = m_0 e^{\phi/c^2}$ and avoids the undesired restriction $d\phi/dt = 0$ of the theory presented before⁶. The matter evolution equation can thus be written as

$$\frac{du_\mu}{d\tau} = -\partial_\mu \phi - \frac{d\phi}{d\tau} u_\mu. \quad (1.4)$$

It is worth noting that this equation satisfies Newton's equivalence principle in the sense that the gravitational mass of a body is identified with its rest mass. Free fall, therefore, turns out to be independent of the rest mass of the body. However, Einstein's special theory of relativity had shown a deep relation between mass and energy that should be carefully addressed in the construction of any relativistic theory of gravity. The equation $E = mc^2$, where $m = \gamma m_0$ and $\gamma = 1/\sqrt{1 - \vec{v}^2/c^2}$, states that kinetic energy increases the effective mass of a body, its inertia. Therefore, if inertial mass is the source of the gravitational field, a moving body could generate a stronger gravitational field than the same body at rest. By extension of this reasoning, one can conclude that bodies with different internal energies could fall differently in an external gravitational field. Einstein found this point disturbing and used it to criticize Nordström's theory. In addition, in this theory the gravitational potential ϕ of point particles goes to $-\infty$ at the location of the particle, thus implying that point particles are massless and, therefore, cannot exist. One is thus led to consider extended (or continuous) objects, which possess other types of inertia in the form of stresses that cannot be reduced to a mass. The source of the gravitational field, the right hand side of (1.1), should thus take into account also such stresses.

To overcome those problems and others concerning energy conservation pointed out by Einstein, Nordström proposed a second theory [Nordström, 1913]

$$\square \phi = g(\phi)\nu \quad (1.5)$$

$$\mathcal{F}_\mu = -g(\phi)\nu \partial_\mu \phi. \quad (1.6)$$

where \mathcal{F} represents the force per unit volume and $g(\phi)\nu$ is a density that represents the source of the gravitational field. To determine the functional form

⁶Varying speed of light theories may also avoid the restriction $d\phi/d\tau = 0$, but such theories break the essence of special relativity by definition.

of $g(\phi)$ and find a natural correspondence between ν and the matter sources, Nordström proceeded as follows. Firstly, he defined the *gravitational mass* of a system using the right hand side of (1.5) and (1.6) as

$$M_g = \int d^3x g(\phi) \nu . \quad (1.7)$$

Then he assumed that the *inertial mass* of the system should be a Lorentz scalar made out of all the energy sources, which include the rest mass and stresses associated to the matter, the gravitational field, and the electromagnetic field. He thus proposed the following expression

$$m_i = -\frac{1}{c^2} \int d^3x [T_m + G_\phi + F_{em}] , \quad (1.8)$$

where the trace of the stress-energy tensor of the matter is represented by T_m , the trace of the electromagnetic field by F_{em} (which vanishes), and that of the gravitational field by G_m , being $G_{\mu\nu} = (2/\kappa^2)[\partial_\mu\phi\partial_\nu\phi - (1/2)\eta_{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi)]$ the stress-energy tensor of the (scalar) gravitational field.

To force the equivalence between inertial and gravitational mass in a system of particles immersed in an external gravitational field with potential ϕ_a , Nordström imposed that for such a system the following relation should hold

$$M_g = g(\phi_a) m_i . \quad (1.9)$$

Then he considered a stationary system on that gravitational field and showed that the contribution of the local gravitational field to the total inertia of the system was given by

$$-\frac{1}{c^2} \int d^3x G_\phi = -\frac{1}{c^2} \int d^3x (\phi - \phi_a) g(\phi) \nu . \quad (1.10)$$

Combining this expression with (1.9) and (1.8) one finds that

$$\int d^3x \left[T_m + g(\phi) \nu \left(\phi - \phi_a + \frac{c^2}{g(\phi_a)} \right) \right] = 0 . \quad (1.11)$$

Demanding proportionality between T_m and ν , one finds that $g(\phi) = C/(A + \phi)$. A natural gauge corresponds to $g(\phi) = -4\pi G/\phi$ because it allows to recover the Newtonian result $E_0 = mc^2 = M_g \phi_a$ that implies that the energy of a system with gravitational mass M_g in a field with potential ϕ_a is exactly $M_g \phi_a$. Therefore, from Nordström's second theory it follows that the inertial mass of a stationary system varies in proportion to the external potential whereas M_g remains constant, i.e., $m/\phi = \text{constant}$.

With the above results one finds that (1.5) and (1.6) turn into (from now on $\kappa^2 \equiv 8\pi G$)

$$\phi \square \phi = -\frac{\kappa^2}{2} T_m \quad (1.12)$$

$$\frac{du_\mu}{d\tau} = -\partial_\mu \ln \phi - u_\mu \frac{d}{d\tau} \ln \phi . \quad (1.13)$$

Using these equations it is straightforward to verify that the total energy-momentum of the system is conserved, i.e., $\partial^\mu (T_{\mu\nu}^\phi + T_{\mu\nu}^m) = 0$, where one must take $T_{\mu\nu}^m = \rho \phi u_\mu u_\nu$ for pressureless matter because, as shown above, the inertial rest mass density of a system grows linearly with ϕ .

Nordström's second theory, therefore, represents a satisfactory example of relativistic theory of gravity in Minkowski space that satisfies the equivalence between inertial and gravitational mass and in which energy and momentum are conserved. Unfortunately, it does not predict any bending of light and also fails in other predictions that were important at the beginning of the twentieth century such as the perihelion shift of Mercury. Nonetheless, it admits a geometric interpretation that greatly simplifies its structure and puts forward the direction in which Einstein's work was progressing.

Considering a line element of the form $ds^2 = \phi^2(-dt^2 + d\vec{x}^2)$, Einstein and Fokker showed that the matter evolution equation (1.13) could be obtained by extremizing the path followed by a free particle in that geometry, i.e., by computing the variation $\delta(-mc^2 \int ds) = 0$ [Einstein and Fokker, 1914]. This variation yields the geodesic equation⁷

$$\frac{d\tilde{u}^\mu}{d\tilde{\tau}} + \Gamma_{\alpha\beta}^\mu \tilde{u}^\alpha \tilde{u}^\beta = 0 , \quad (1.14)$$

where $\Gamma_{\alpha\beta}^\mu = \partial_\alpha \phi \delta_\beta^\mu + \partial_\beta \phi \delta_\alpha^\mu - \eta^{\mu\rho} \partial_\rho \phi \eta_{\alpha\beta}$. The gravitational field equation also takes a very interesting form

$$R = 3\kappa^2 \tilde{T}_m , \quad (1.15)$$

where $R = -(6/\phi^3) \eta^{\alpha\beta} \partial_\alpha \partial_\beta \phi$ and $\tilde{T}_m = T_m/\phi^4$ due to the conformal transformation that relates the background metric with the Minkowski metric. These last results represent generally covariant equations that establish a non-trivial relation between gravitation and geometry. Though this theory was eventually ruled out by observations, its potential impact on the eventual formal and conceptual formulation of Einstein's general theory of relativity must have been important.

1.2.2 To general relativity via general covariance

The Principle of Relativity together with Newton's ideas about the equivalence between inertial and gravitational mass led Einstein to develop what has come to be called the Einstein equivalence principle (EEP), which will be introduced later in detail. Einstein wanted to extend the principle of relativity not only to inertial observers (special relativity) but to all kinds of motion (hence the term general relativity). This motivated the search for generally covariant equations

⁷To obtain (1.13) from the geodesic equation one should note that $d\tilde{\tau} = \phi d\tau$, $\tilde{u}_\mu = \phi u_\mu$, and that the indices in (1.13) are raised and lowered with $\eta_{\mu\nu}$.

Though it is not difficult to realize that one can construct a fully covariant theory in Minkowski space, the consideration of arbitrary accelerated frames leads to the appearance of inertial or fictitious forces whose nature is difficult to interpret. This is due to the fact that Minkowski spacetime, like Newtonian space, is an absolute space. The possibility of writing the laws of physics in a coordinate (cartesian, polar, ...) and frame (inertial, accelerated, ...) invariant way, helped Einstein to realize that a local, homogeneous gravitational field is indistinguishable from a constant acceleration. This allowed him to introduce the concept of local inertial frame (LIF) and find a correspondence between gravitation and geometry, which led to a deep conceptual change: **there exists no absolute space**. This follows from the fact that, unlike other well-known forces, the local effects of gravity can always be eliminated by a suitable choice of coordinates (Einstein's elevator).

The forces of Newtonian mechanics, which were thought to be measured with respect to *absolute space*, were in fact being measured in an accelerated frame (static with respect to the Earth), which led to the appearance of the observed gravitational acceleration. According to Einstein, accelerations produced by interactions such as the electromagnetic field should be measured in LIFs. This means that they should be measured not with respect to absolute space but with respect to the local gravitational field (which defines LIFs). In other words, Einstein identified the Newtonian absolute space with the local gravitational field. Physical accelerations should, therefore, be measured in local inertial frames, where Minkowskian physics should be recovered. Gravitation, according to Einstein, was intrinsically different from the rest of interactions. It was a geometrical phenomenon.

The geometrical interpretation of gravitation implied that it should be described by a tensor field, the metric $g_{\mu\nu}$, which boils down to the Minkowski metric locally in appropriate coordinate systems (LIFs) or globally when gravitation is absent. This view made it natural to interpret the effects of a gravitational field on particles as geodesic motion. In the absence of non-gravitational interactions, particles should follow geodesics of the background metric, which are formally described by eq.(1.14) but with $\Gamma_{\alpha\beta}^{\mu}$, the so-called Levi-Civita connection, defined in terms of a symmetric metric tensor $g_{\mu\nu}$ as

$$\Gamma_{\alpha\beta}^{\mu} = \frac{g^{\mu\rho}}{2} [\partial_{\alpha}g_{\rho\beta} + \partial_{\beta}g_{\rho\alpha} - \partial_{\rho}g_{\alpha\beta}] . \quad (1.16)$$

To determine the dynamics of the metric tensor one needs at least ten independent equations, as many as independent components there are in $g_{\mu\nu}$. Since the source of the gravitational field must be related with the stress-energy tensor of matter and the dynamics of classical mechanics is generally governed by second-

⁸The idea of general covariance is nowadays naturally seen as a basic mathematical requirement in any theory based on the use of differential manifolds. In this sense, though general covariance forces the use of tensor calculus, it should be noted that it does not necessarily imply curved space-time. Note also that it is the connection, not the metric, the most important object in the construction of tensors.

order equations, Einstein proposed the following set of tensorial equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu} , \quad (1.17)$$

where $R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$ is the so-called Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and $R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}$ represents the components of the Riemann tensor, the field strength of the connection $\Gamma^\alpha_{\mu\beta}$, which here is defined as in (1.16).

Eq. (1.17) represents a system of non-linear, second-order partial differential equations for the ten independent components of the metric tensor. The conservation of energy and momentum is guaranteed independently for the left and the right hand sides of (1.17). The contraction⁹ $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$ follows from a geometrical identity, whereas $\nabla^\mu T_{\mu\nu} = 0$ follows if the Minkowski equations of motion for the matter fields are satisfied locally. The non-linearity of the equations manifests the fact that the energy stored in the gravitational field can source the gravitational field itself in a non-trivial way. Unlike Nordström's second theory, this set of tensorial equations imply that the gravitational field is sourced by the full stress-energy tensor, not just by its trace. This implies that electromagnetic fields, like any other matter sources, generate a non-zero Ricci tensor and, therefore, gravitate.

Einstein's theory was rapidly accepted despite its poor experimental verification. In fact, we had to wait until the 1960's to have the perihelion shift of Mercury and the deflection of light by the sun measured to within an accuracy of $\sim 1\%$ and $\sim 50\%$, respectively. In 1959 Pound and Rebka were able to measure the gravitational redshift for the first time. Additionally, though Hubble's discoveries on the recession of distant galaxies had boosted Einstein's popularity, those observations were a mere qualitative verification of the effect and only recently has it been possible to contrast theory and observations with some confidence in the cosmological setting. It is therefore not surprising that between 1905 and 1960, there appeared at least 25 alternative relativistic theories of gravitation, where spacetime was flat and gravitation was a Lorentz-invariant field on that background. Though many researchers defended Einstein's idea of curved spacetime, others like Birkhoff did not [Birkhoff, 1944]:

The initial attempts to incorporate gravitational phenomena in flat space-time were not satisfactory. Einstein turned to the curved spacetime suggested by his principle of equivalence, and so constructed his general theory of relativity. The initial predictions, based on this celebrated theory of gravitation, were brilliantly confirmed. However, the theory has not led to any further applications and, because of its complicated mathematical character, seems to be essentially unworkable. Thus curved spacetime has come to be regarded by many as an auxiliary construct (Larmor) rather than as a physical reality.

Such strong claims suggest that it was necessary a careful analysis of the foundations of Einstein's theory: is spacetime really curved or is gravitation

⁹The differential operator ∇_μ represents a covariant derivative, which is the natural extension of the usual flat space derivative ∂_μ to spaces with non-trivial parallel transport.

a tensor-like interaction in a flat background? The next section is devoted to clarify these points and others that will help establish the foundations of gravitation theory.

1.2.3 The Einstein equivalence principle

The experimental facts that support the foundations of gravitation should never be underestimated since they provide a valuable guide in the construction of viable theories and in constraining the realm of speculation. In this sense, the experimental efforts carried out by Robert Dicke in the 1960's [Dicke, 1964] resulted in what has come to be called the Einstein equivalence principle (EEP) and constitute a fundamental pillar for gravitation theory. We will briefly review next the experimental evidence supporting it, and the way it enters in the construction of gravitation theories [Will, 1993]. The EEP states that [Will, 2005]

- Inertial and gravitational masses coincide (weak equivalence principle).
- The outcome of any non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed (Local Lorentz Invariance).
- The outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed (Local Position Invariance).

Let us briefly discuss the experimental evidence supporting the EEP.

Weak equivalence principle

A direct test of WEP is the comparison of the acceleration of two laboratory-sized bodies of different composition in an external gravitational field. If the principle were violated, then the accelerations of different bodies would differ. In Dicke's torsion balance experiment, for instance, the gravitational acceleration toward the sun of small gold and aluminum weights were compared and found to be equal with an accuracy of about a part in 10^{11} . One should note that gold and aluminum atoms have very different properties, which is important for testing how gravitation couples to different particles and interactions. For instance, the electrons in aluminum are non-relativistic whereas the k-shell electrons of gold have a 15% increase in their mass as a result of their relativistic velocities. The electromagnetic negative contribution to the binding energy of the nucleus varies as Z^2 and represents 0.5% of the total mass of a gold atom, whereas it is negligible in Al. Additionally, the virtual pair field, pion field, etcetera, around the gold nucleus would be expected to represent a far bigger contribution to the total energy than in aluminum. This makes it clear that a gold sphere possesses additional inertial contributions due to the electromagnetic, weak, and strong interactions that are not present (or are negligible) in the aluminum sphere. If any of those sources of inertia did not contribute by the same amount to the gravitational mass of the system, then gold and aluminum would fall with different accelerations.

The precision of Dicke's experiment was such that from it one can conclude, for instance, that positrons and other antiparticles fall down, not up [Dicke, 1964]. This is so because if the positrons in the pair field of the gold atom were to tend to fall up, not down, there would be an anomalous weight of the atom substantially greater for large atomic number than small.

Tests of local Lorentz invariance

The existence of a preferred reference frame breaking the local isotropy of space would imply a dependence of the speed of light on the direction of propagation. This would cause shifts in the energy levels of atoms and nuclei that depend on the orientation of the quantization axis of the state relative to our universal velocity vector, and on the quantum numbers of the state. This idea was tested by Hughes (1960) and Drever (1961), who examined the $J = 3/2$ ground state of the ${}^7\text{Li}$ nucleus in an external magnetic field. If the Michelson-Morley experiment had found $\delta \equiv c^{-2} - 1 \approx 10^{-3}$, the Hughes-Drever experiment set the limit to $\delta \approx 10^{-15}$. More recent experiments using laser-cooled trapped atoms and ions have reached $\delta \approx 10^{-17}$.

Currently, new ideas coming from quantum gravity (with a minimal length scale), braneworld scenarios, and models of string theory have motivated new ways to test Lorentz invariance by considering Lorentz-violating parameters in extensions of the standard model and also some astrophysical tests. So far, however, no compelling evidence for a violation of Lorentz invariance has been found.

Tests of local position invariance

Local position invariance can be tested by gravitational redshift experiments, which test the existence of spatial dependence on the outcome of local non-gravitational experiments, and by measurements of the fundamental non-gravitational constants that test for temporal dependence. Gravitational redshift experiments usually measure the frequency shift $Z = \Delta\nu/\nu = -\Delta\lambda/\lambda$ between two identical frequency standards (clocks) placed at rest at different heights in a static gravitational field. If the frequency of a given type of atomic clock is the same when measured in a local, momentarily comoving freely falling frame (Lorentz frame), independent of the location or velocity of that frame, then the comparison of frequencies of two clocks at rest at different locations boils down to a comparison of the velocities of two local Lorentz frames, one at rest with respect to one clock at the moment of emission of its signal, the other at rest with respect to the other clock at the moment of reception of the signal. The frequency shift is then a consequence of the first-order Doppler shift between the frames. The result is a shift $Z = \frac{\Delta U}{c^2}$, where U is the difference in the Newtonian gravitational potential between the receiver and the emitter. If the frequency of the clocks had some dependence on their position, the shift could be written as $Z = (1 + \alpha) \frac{\Delta U}{c^2}$. Comparison of a hydrogen-maser clock flown on a rocket to an altitude of about 10.000 km with a similar clock on the ground yielded a limit

$$\alpha < 2 \times 10^{-4}.$$

Another important aspect of local position invariance is that if it is satisfied then the fundamental constants of non-gravitational physics should be constants in time. Though these tests are subject to many uncertainties and experimental limitations, there is no strong evidence for a possible spatial or temporal dependence of the fundamental constants.

1.2.4 Metric theories of gravity

The EEP is not just a verification that gravitation can be associated with a metric tensor which locally can be turned into the Minkowskian metric by a suitable choice of coordinates. If it is valid, then gravitation must be a curved space-time phenomenon, i.e., the effects of gravity must be equivalent to the effects of living in a curved space-time. For this reason, the only theories of gravity that have a hope of being viable are those that satisfy the following postulates (see [Will, 1993] and [Will, 2005]):

1. Spacetime is endowed with a symmetric metric.
2. The trajectories of freely-falling bodies are geodesics of that metric.
3. In local freely-falling reference frames, the non-gravitational laws of physics are those written in the language of special relativity.

Theories satisfying these postulates are known as *metric theories of gravity*, and their action can be written generically as

$$S_{MT} = S_G[g_{\mu\nu}, \phi, A_\mu, B_{\mu\nu}, \dots] + S_m[g_{\mu\nu}, \psi_m], \quad (1.18)$$

where $S_m[g_{\mu\nu}, \psi_m]$ represents the matter action, ψ_m denotes the matter and non-gravitational fields, and S_G is the gravitational action, which besides the metric $g_{\mu\nu}$ may depend on other gravitational fields (scalars, vectors, and tensors of different ranks). This form of the action guarantees that the non-gravitational fields of the standard model of elementary particles couple to gravitation only through the metric, which should allow to recover locally the non-gravitational physics of Minkowski space. The construction of $S_m[g_{\mu\nu}, \psi_m]$ can thus be carried out by just taking its Minkowski space form and going over to curved space-time using the methods of differential geometry. It should be noted that the EEP does neither point towards GR as the preferred theory of gravity nor provides any constraint or hint on the functional form of the gravitational part of the action. The functional S_G must provide dynamical equations for the metric (and the other gravitational fields, if there are any) but its form must be obtained by theoretical reasoning and/or by experimental exploration.

It is worth noting that if S_G contains other long-range fields besides the metric, then **gravitational** experiments in a local, freely falling frame may depend on the location and velocity of the frame relative to the external environment. This is so because, unlike the metric, the boundary conditions induced by those

fields cannot be trivialized by a suitable choice of coordinates. Of course, local **non-gravitational** experiments are unaffected since the gravitational fields they generate are assumed to be negligible, and since those experiments couple only to the metric, whose form can always be made locally Minkowskian at a given spacetime event.

Before concluding this section, it should be noted that string theories predict the existence of new kinds of fields with couplings to fermions and the interactions of the standard model in a way that breaks the simplicity of metric theories of gravity, i.e., they do not allow for a clean splitting of the action into a matter sector plus a gravitational sector. Such theories, therefore, must be regarded as non-metric. Improved tests of the EEP could be used to test the presence and/or intensity of such couplings, which are expected to represent short range interactions. These tests can be seen as a branch of high-energy physics not based on particle accelerators.

Two examples of metric theories: General relativity and Brans-Dicke theory.

The field equations of Einstein's theory of general relativity (GR) can be derived from the following action

$$S[g_{\mu\nu}, \psi_m] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R(g) + S_m[g_{\mu\nu}, \psi_m] \quad (1.19)$$

where R is the Ricci scalar defined below eq.(1.17). Variation of this action with respect to the metric leads to Einstein's field equations¹⁰

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (1.20)$$

In Einstein's theory, gravity is mediated by a rank-2 tensor field, the metric, and curvature is generated by the matter sources. Brans-Dicke theory introduces, besides the metric, a new gravitational field, which is a scalar. This scalar field is coupled to the curvature as follows

$$S[g_{\mu\nu}, \phi, \psi_m] = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R(g) - \frac{\omega}{\phi} (\partial_\mu \phi \partial^\mu \phi) - V(\phi) \right] + S_m[g_{\mu\nu}, \psi_m] \quad (1.21)$$

In the original Brans-Dicke theory, the potential was set to zero, $V(\phi) = 0$, so the theory had only one free parameter, the constant ω in front of the kinetic energy term, which had to be determined experimentally. Note that the Brans-Dicke scalar has the same dimensions as the inverse of Newton's constant and, therefore, can be seen as related to it. In Brans-Dicke theory, one can thus say that Newton's constant is no longer constant but is, in fact, a dynamical field. The field equations for the metric are

$$R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = \frac{8\pi}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} [\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi] + \frac{\omega}{\phi^2} \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right] \quad (1.22)$$

¹⁰Recall that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and that $\delta R_{\mu\nu} = -\nabla_\mu \delta\Gamma_{\lambda\nu}^\lambda + \nabla_\lambda \delta\Gamma_{\mu\nu}^\lambda$.

The equation that governs the scalar field is

$$(3 + 2\omega)\Box\phi + 2V(\phi) - \phi\frac{dV}{d\phi} = \kappa^2 T \quad (1.23)$$

In this theory we observe that both the matter and the scalar field act as sources for the metric, which means that both the matter and the scalar field generate the spacetime curvature. In fact, even in vacuum the scalar field curves the spacetime. According to the way we wrote the metric field equations, it is tempting to identify the Brans-Dicke field with a new matter field. However, since the Brans-Dicke scalar is sourced by the energy-momentum tensor (via its trace, which is a scalar magnitude constructed out of the sources of energy and momentum), we say that it is a gravitational field. Note, in this sense, that standard matter fields, such as a Dirac field coupled to electromagnetism ($i\gamma^\mu\partial_\mu - m)\psi = e\gamma^\mu A_\mu\psi$, do not couple to energy and momentum.

1.3 Experimental determination of the gravity Lagrangian

Einstein's theory of general relativity (GR) represents one of the most impressive exercises of human intellect. As we have seen in previous sections, it implied a huge conceptual jump with respect to Newtonian gravity and, unlike the currently established standard model of elementary particles, no experiments were carried out to probe the structure of the theory. In spite of that, to date the theory has successfully passed all precision experimental tests. Its predictions are in agreement with experiments in scales that range from millimeters to astronomical units, scales in which weak and strong field phenomena can be observed [Will, 2005]. The theory is so successful in those regimes and scales that it is generally accepted that it should also work at larger and shorter scales, and at weaker and stronger regimes.

This extrapolation is, however, forcing us today to draw a picture of the universe that is not yet supported by other independent observations. For instance, to explain the rotation curves of spiral galaxies, we must accept the existence of vast amounts of unseen matter surrounding those galaxies. Additionally, to explain the luminosity-distance relation of distant type Ia supernovae and some properties of the distribution of matter and radiation at large scales, we must accept the existence of yet another source of energy with repulsive gravitational properties (see [Copeland et al., 2006], [Padmanabhan, 2003], [Peebles and Ratra, 2003] for recent reviews on dark energy). Together those unseen (or dark) sources of matter and energy are found to make up to 96% of the total energy of the observable universe! This huge discrepancy between the gravitationally estimated amounts of matter and energy and the direct measurements via electromagnetic radiation motivates the search for alternative theories of gravity which can account for the large scale dynamics and structure without the need for dark matter and/or dark energy.

In this sense, there has been an enormous international effort in the last years to determine whether the gravity Lagrangian could depart from Einstein's one at cosmic scales in a way compatible with the cosmological observations that support the cosmic speedup. In particular, many authors have investigated the consequences of promoting the Hilbert-Einstein Lagrangian to an arbitrary function $f(R)$ of the scalar curvature (see [Olmo, 2011a], [De Felice and Tsujikawa, 2010], [Sotiriou and Faraoni, 2010], [Capozziello and Francaviglia, 2008] for recent reviews). In this section we will show that the dynamics of the solar system can be used to set important constraints on the form of the function $f(R)$.

1.3.1 Field equations of $f(R)$ theories.

The action that defines $f(R)$ theories has the generic form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m[g_{\mu\nu}, \psi_m] , \quad (1.24)$$

where $\kappa^2 = 8\pi G$, and we use the same notation introduced in previous sections. Variation of (1.24) leads to the following field equations for the metric

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + g_{\mu\nu} \square f_R = \kappa^2 T_{\mu\nu} \quad (1.25)$$

where $f_R \equiv df/dR$. According to (1.25), we see that, in general, the metric satisfies a system of fourth-order partial differential equations. The trace of (1.25) takes the form

$$3\square f_R + R f_R - 2f = \kappa^2 T \quad (1.26)$$

If we take $f(R) = R - 2\Lambda$, (1.25) boils down to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} - \Lambda g_{\mu\nu} , \quad (1.27)$$

which represents GR with a cosmological constant. This is the only case in which an $f(R)$ Lagrangian yields second-order equations for the metric¹¹.

Let us now rewrite (1.25) in the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{f_R} T_{\mu\nu} - \frac{1}{2f_R} g_{\mu\nu} [R f_R - f] + \frac{1}{f_R} [\nabla_\mu \nabla_\nu f_R - g_{\mu\nu} \square f_R] \quad (1.28)$$

The right hand side of this equation can now be seen as the source terms for the metric. This equation, therefore, tells us that the metric is generated by the matter and by terms related to the scalar curvature. If we now wonder about what generates the scalar curvature, the answer is in (1.26). That expression

¹¹This is so only if the connection is assumed to be the Levi-Civita connection of the metric (metric formalism). If the connection is regarded as independent of the metric, Palatini formalism, then $f(R)$ theories lead to second-order equations. This point will be explained in detail later on.

says that the scalar curvature satisfies a second-order differential equation with the trace T of the energy-momentum tensor of the matter and other curvature terms acting as sources. We have thus clarified the role of the higher-order derivative terms present in (1.25). The scalar curvature is now a dynamical entity which helps generate the space-time metric and whose dynamics is determined by (1.26).

At this point one should have noted the essential difference between a generic $f(R)$ theory and GR. In GR the only dynamical field is the metric and its form is fully characterized by the matter distribution through the equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The scalar curvature is also determined by the local matter distribution but through an algebraic equation, namely, $R = -\kappa^2 T$. In the $f(R)$ case both $g_{\mu\nu}$ and R are dynamical fields, i.e., they are governed by differential equations. Furthermore, the scalar curvature R , which can obviously be expressed in terms of the metric and its derivatives, now plays a non-trivial role in the determination of the metric itself.

The physical interpretation given above puts forward the central and active role played by the scalar curvature in the field equations of $f(R)$ theories. However, (1.26) suggests that the actual dynamical entity is f_R rather than R itself. This is so because, besides the metric, f_R is the only object acted on by differential operators in the field equations. Motivated by this, we can introduce the following alternative notation

$$\phi \equiv f_R \quad (1.29)$$

$$V(\phi) \equiv R(\phi)f_R - f(R(\phi)) \quad (1.30)$$

and rewrite eqs. (1.28) and (1.26) in the same form as (1.22) and (1.23) with the choice $w = 0$. This slight change of notation helps us identify the $f(R)$ theory in metric formalism with a scalar-tensor Brans-Dicke theory with parameter $\omega = 0$ and non-trivial potential $V(\phi)$, whose action was given in (1.21). In terms of this scalar-tensor representation our interpretation of the field equations of $f(R)$ theories is obvious, since both the matter and the scalar field help generate the metric. The scalar field is a dynamical object influenced by the matter and by self-interactions according to (1.23).

1.3.2 Spherically symmetric systems

A complete description of a physical system must take into account not only the system but also its interaction with the environment. In this sense, any physical system is surrounded by the rest of the universe. The relation of the local system with the rest of the universe manifests itself in a set of boundary conditions. In our case, according to (1.26) and (1.28), the metric and the function f_R (or, equivalently, R or ϕ) are subject to boundary conditions, since they are dynamical fields (they are governed by differential equations). The boundary conditions for the metric can be trivialized by a suitable choice of coordinates. In other words, we can make the metric Minkowskian in the asymptotic region and fix its first derivatives to zero (see chapter 4 of [Will, 1993] for details). The

function f_R , on the other hand, should tend to the cosmic value f_{R_c} as we move away from the local system. The precise value of f_{R_c} is obtained by solving the equations of motion for the corresponding cosmology. According to this, the local system will interact with the asymptotic (or background) cosmology via the boundary value f_{R_c} and its cosmic-time derivative. Since the cosmic time-scale is much larger than the typical time-scale of local systems (billions of years versus years), we can assume an adiabatic interaction between the local system and the background cosmology. We can thus neglect terms such as \dot{f}_{R_c} , where dot denotes derivative with respect to the cosmic time.

The problem of finding solutions for the local system, therefore, reduces to solving (1.28) expanding about the Minkowski metric in the asymptotic region¹², and (1.26) tending to

$$3\Box_c f_{R_c} + R_c f_{R_c} - 2f(R_c) = \kappa^2 T_c \quad (1.31)$$

where the subscript c denotes cosmic value, far away from the system. In particular, if we consider a weakly gravitating local system, we can take $f_R = f_{R_c} + \varphi(x)$ and $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|\varphi| \ll |f_{R_c}|$ and $|h_{\mu\nu}| \ll 1$ satisfying $\varphi \rightarrow 0$ and $h_{\mu\nu} \rightarrow 0$ in the asymptotic region. Note that should the local system represent a strongly gravitating system such as a neutron star or a black hole, the perturbative expansion would not be sufficient everywhere. In such cases, the perturbative approach would only be valid in the far region. Nonetheless, the decomposition $f_R = f_{R_c} + \varphi(x)$ is still very useful because the equation for the local deviation $\varphi(x)$ can be written as

$$3\Box\varphi + W(f_{R_c} + \varphi) - W(f_{R_c}) = \kappa^2 T, \quad (1.32)$$

where T represents the trace of the local sources, we have defined $W(f_R) \equiv R(f_R)f_R - 2f(R[f_R])$, and $W(f_{R_c})$ is a slowly changing constant within the adiabatic approximation. In this case, φ needs not be small compared to f_{R_c} everywhere, only in the asymptotic regions.

Spherically symmetric solutions

Let us define the line element¹³ [Olmo, 2007]

$$ds^2 = -A(r)e^{2\psi(r)}dt^2 + \frac{1}{A(r)}(dr^2 + r^2d\Omega^2), \quad (1.33)$$

which, assuming a perfect fluid for the sources, leads to the following field equations

$$A_{rr} + A_r \left[\frac{2}{r} - \frac{5}{4} \frac{A_r}{A} \right] = \frac{\kappa^2 \rho}{f_R} + \frac{Rf_R - f(R)}{2f_R} + \frac{A}{f_R} \left[f_{Rrr} + f_{Rr} \left(\frac{2}{r} - \frac{A_r}{2A} \right) \right] \quad (1.34)$$

¹²Note that the expansion about the Minkowski metric does not imply the existence of global Minkowskian solutions. As we will see, the general solutions to our problem turn out to be asymptotically de Sitter spacetimes.

¹³As pointed out in [Olmo, 2007], solar system tests are conventionally described in isotropic coordinates rather than on Schwarzschild-like coordinates. This justifies our coordinate choice in (1.33).

$$A\psi_r \left[\frac{2}{r} + \frac{f_{Rr}}{f_R} - \frac{A_r}{A} \right] - \frac{A_r^2}{4A} = \frac{\kappa^2 P}{f_R} - \frac{Rf_R - f(R)}{2f_R} - A \frac{f_{Rr}}{f_R} \left[\frac{2}{r} - \frac{A_r}{2A} \right] \quad (1.35)$$

where $f_R = f_{R_c} + \varphi$, and the subscripts r in ψ_r , f_{Rr} , f_{Rrr} , M_r denote derivation with respect to the radial coordinate. Note also that $f_{Rr} = \varphi_r$ and $f_{Rrr} = \varphi_{rr}$. The equation for φ is, according to (1.32) and (1.33),

$$A\varphi_{rr} = -A \left(\frac{2}{r} + \psi_r \right) \varphi_r - \frac{W(f_{R_c} + \varphi) - W(f_{R_c})}{3} + \frac{\kappa^2}{3}(3P - \rho) \quad (1.36)$$

Equations (1.34), (1.35), and (1.36) can be used to work out the metric of any spherically symmetric system subject to the asymptotic boundary conditions discussed above. For weak sources, such as non-relativistic stars like the sun, it is convenient to expand them assuming $|\varphi| \ll f_{R_c}$ and $A = 1 - 2M(r)/r$, with $2M(r)/r \ll 1$. The result is

$$-\frac{2}{r}M_{rr}(r) = \frac{\kappa^2 \rho}{f_{R_c}} + V_c + \frac{1}{f_{R_c}} \left[\varphi_{rr} + \frac{2}{r}\varphi_r \right] \quad (1.37)$$

$$\frac{2}{r} \left[\psi_r + \frac{\varphi_r}{f_{R_c}} \right] = \frac{\kappa^2}{f_{R_c}} P - V_c \quad (1.38)$$

$$\varphi_{rr} + \frac{2}{r}\varphi_r - m_c^2 \varphi = \frac{\kappa^2}{3}(3P - \rho) \quad (1.39)$$

where we have defined

$$V_c \equiv \left. \frac{Rf_R - f}{2f_R} \right|_{R_c} \quad \text{and} \quad m_c^2 \equiv \left. \frac{f_R - Rf_{RR}}{3f_{RR}} \right|_{R_c}. \quad (1.40)$$

This expression for m_c^2 was first found in [Olmo, 2005a, Olmo, 2005b] within the scalar-tensor approach. It was found there that $m_c^2 > 0$ is needed to have a well-behaved (non-oscillating) Newtonian limit. This expression and the conclusion $m_c^2 > 0$ were also reached in [Faraoni and Nadeau, 2005] by studying the stability of de Sitter space. The same expression has been rediscovered later several times.

Outside of the sources, the solutions of (1.37), (1.38) and (1.39) lead to

$$\varphi(r) = \frac{C_1}{r} e^{-m_c r} \quad (1.41)$$

$$A(r) = 1 - \frac{C_2}{r} \left(1 - \frac{C_1}{C_2 f_{R_c}} e^{-m_c r} \right) + \frac{V_c}{6} r^2 \quad (1.42)$$

$$A(r) e^{2\psi} = 1 - \frac{C_2}{r} \left(1 + \frac{C_1}{C_2 f_{R_c}} e^{-m_c r} \right) - \frac{V_c}{3} r^2 \quad (1.43)$$

where an integration constant ψ_0 has been absorbed in a redefinition of the time coordinate. The above solutions coincide, as expected, with those found in [Olmo, 2005a, Olmo, 2005b] for the Newtonian and post-Newtonian limits

using the scalar-tensor representation and standard gauge choices in Cartesian coordinates. Comparing our solutions with those, we identify

$$C_2 \equiv \frac{\kappa^2}{4\pi f_{R_c}} M_\odot \quad \text{and} \quad \frac{C_1}{f_{R_c} C_2} \equiv \frac{1}{3} \quad (1.44)$$

where $M_\odot = \int d^3x \rho(x)$. The line element (1.33) can thus be written as

$$ds^2 = - \left(1 - \frac{2GM_\odot}{r} - \frac{V_c}{3} r^2 \right) dt^2 + \left(1 + \frac{2G\gamma M_\odot}{r} - \frac{V_c}{6} r^2 \right) (dr^2 + r^2 d\Omega^2) \quad (1.45)$$

where we have defined the effective Newton's constant and post-Newtonian parameter γ as

$$G = \frac{\kappa^2}{8\pi f_{R_c}} \left(1 + \frac{e^{-m_c r}}{3} \right) \quad \text{and} \quad \gamma = \frac{3 - e^{-m_c r}}{3 + e^{-m_c r}} \quad (1.46)$$

respectively. This completes the lowest-order solution in isotropic coordinates.

The gravity Lagrangian according to solar system experiments

From the definitions of Eq.(1.46) we see that the parameters G and γ that characterize the linearized metric depend on the effective mass m_c (or inverse length scale $\lambda_{m_c} \equiv m_c^{-1}$). Newton's constant, in addition, also depends on f_{R_c} . Since the value of the background cosmic curvature R_c changes with the cosmic expansion, it follows that f_{R_c} and m_c must also change. The variation in time of f_{R_c} induces a time variation in the effective Newton's constant which is just the well-known time dependence that exists in Brans-Dicke theories. The length scale λ_{m_c} , characteristic of $f(R)$ theories, does not appear in the original Brans-Dicke theories because in the latter the scalar potential was assumed to vanish, $V(\phi) \equiv 0$, in contrast with (1.30), which implies an infinite interaction range ($m_c = 0 \rightarrow \lambda_{m_c} = \infty$).

In order to have agreement with the observed properties of the solar system, the Lagrangian $f(R)$ must satisfy certain basic constraints. These constraints will be very useful to determine the viability of some families of models proposed to explain the cosmic speedup. A very representative family of such models, which do exhibit self-accelerating late-time cosmic solutions, is given by $f(R) = R - R_0^{n+1}/R^n$, where R_0 is a very low curvature scale that sets the scale at which the model departs from GR, and n is assumed positive. At curvatures higher than R_0 , the theory is expected to behave like GR while at late times, when the cosmic density decays due to the expansion and approaches the scale R_0 , the modified dynamics becomes important and could explain the observed speedup.

In viable theories, the effective cosmological constant V_c must be negligible. Most importantly, the interaction range λ_{m_c} must be shorter than a few millimeters because such Yukawa-type corrections to the Newtonian potential have not been observed, and observations indicate that the parameter γ is very close

to unity. This last constraint can be expressed as $(L_S^2/\lambda_{m_c}^2) \gg 1$, where L_S represents a (relatively short) length scale that can range from meters to planetary scales, depending on the particular test used to verify the theory. In terms of the Lagrangian, this constraint takes the form

$$\left. \frac{f_R - R f_{RR}}{3 f_{RR}} \right|_{R_c} L_S^2 \gg 1. \quad (1.47)$$

A qualitative analysis of this constraint can be used to argue that, in general, $f(R)$ theories with terms that become dominant at low cosmic curvatures, such as the models $f(R) = R - R_0^{n+1}/R^n$, are not viable theories in solar system scales and, therefore, cannot represent an acceptable mechanism for the cosmic expansion.

Roughly speaking, eq.(1.47) says that the smaller the term $f_{RR}(R_c)$, with $f_{RR}(R_c) > 0$ to guarantee $m_\varphi^2 > 0$, the heavier the scalar field. In other words, the smaller $f_{RR}(R_c)$, the shorter the interaction range of the field. In the limit $f_{RR}(R_c) \rightarrow 0$, corresponding to GR, the scalar interaction is completely suppressed. Thus, if the nonlinearity of the gravity Lagrangian had become dominant in the last few billions of years (at low cosmic curvatures), the scalar field interaction range λ_{m_c} would have increased accordingly. In consequence, gravitating systems such as the solar system, globular clusters, galaxies,... would have experienced (or will experience) observable changes in their gravitational dynamics. Since there is no experimental evidence supporting such a change¹⁴ and all currently available solar system gravitational experiments are compatible with GR, it seems unlikely that the nonlinear corrections may be dominant at the current epoch.

Let us now analyze in detail the constraint given in eq.(1.47). That equation can be rewritten as follows

$$R_c \left[\left. \frac{f_R}{R f_{RR}} \right|_{R_c} - 1 \right] L_S^2 \gg 1 \quad (1.48)$$

We are interested in the form of the Lagrangian at intermediate and low cosmic curvatures, i.e., from the matter dominated to the vacuum dominated eras. We shall now demand that the interaction range of the scalar field remains as short as today or decreases with time so as to avoid dramatic modifications of the gravitational dynamics in post-Newtonian systems with the cosmic expansion. This can be implemented imposing

$$\left[\frac{f_R}{R f_{RR}} - 1 \right] \geq \frac{1}{l^2 R} \quad (1.49)$$

as $R \rightarrow 0$, where $l^2 \ll L_S^2$ represents a bound to the current interaction range of the scalar field. Thus, eq.(1.49) means that the interaction range of the

¹⁴As an example, note that the fifth-force effects of the Yukawa-type correction introduced by the scalar degree of freedom would have an effect on stellar structures and their evolution, which would lead to incompatibilities with current observations.

field must decrease or remain short, $\sim l^2$, with the expansion of the universe. Manipulating this expression, we obtain

$$\frac{d \log[f_R]}{dR} \leq \frac{l^2}{1 + l^2 R} \quad (1.50)$$

which can be integrated twice to give the following inequality

$$f(R) \leq A + B \left(R + \frac{l^2 R^2}{2} \right) \quad (1.51)$$

where B is a positive constant, which can be set to unity without loss of generality. Since f_R and f_{RR} are positive, the Lagrangian is also bounded from below, i.e., $f(R) \geq A$. In addition, according to the cosmological data, $A \equiv -2\Lambda$ must be of order a cosmological constant $2\Lambda \sim 10^{-53} \text{ m}^2$. We thus conclude that the gravity Lagrangian at intermediate and low scalar curvatures is bounded by

$$-2\Lambda \leq f(R) \leq R - 2\Lambda + \frac{l^2 R^2}{2} \quad (1.52)$$

This result shows that a Lagrangian with nonlinear terms that grow with the cosmic expansion is not compatible with the current solar system gravitational tests, such as we argued above. Therefore, those theories cannot represent a valid mechanism to justify the observed cosmic speed-up. Additionally, our analysis has provided an empirical procedure to determine the form of the gravitational Lagrangian. The function $f(R)$ found here nicely recovers Einstein's gravity at low curvatures but allows for some quadratic corrections at higher curvatures, which is of interest in studies of the very early Universe.

1.4 Quantum gravity phenomenology and the early universe

The extrapolation of the dynamics of GR to the very strong field regime indicates that the Universe began at a singularity and that the death of a sufficiently massive star unavoidably leads to the formation of a black hole or a naked singularity. The existence of space-time singularities is one of the most impressive predictions of GR. This prediction, however, also represents the end of the theory, because the absence of a well-defined geometry implies the absence of physical laws and lack of predictability [Novello and Bergliaffa, 2008, Hawking, 1975]. For this reason, it is generally accepted that the dynamics of GR must be changed at some point to avoid these problems. A widespread belief is that at sufficiently high energies the gravitational field must exhibit quantum properties that alter the dynamics and prevent the formation of singularities. However, a completely satisfactory quantum theory of gravity is not yet available. To make some progress in the qualitative understanding of how quantum gravity may affect the dynamics of the Universe near the big bang, in this section we show how certain modifications of GR may be able to capture some aspects of the

expected phenomenology of quantum gravity [Olmo, 2011b].

We begin by noting that Newton's and Planck's constants may be combined with the speed of light to generate a length $l_P = \sqrt{\hbar G/c^3}$, which is known as the Planck length. The Planck length is usually interpreted as the scale at which quantum gravitational phenomena should play a non-negligible role. However, since lengths are not relativistic invariants, the existence of the Planck length raises doubts about the nature of the reference frame in which it should be measured and about the limits of validity of special relativity itself. This poses the following question: can we combine in the same framework the speed of light and the Planck length in such a way that both quantities appear as universal invariants to all observers? The solution to this problem will give us the key to consider quantum gravitational phenomena from a modified gravity perspective.

1.4.1 Palatini approach to modified gravity

To combine in the same framework the speed of light and the Planck length in a way that preserves the invariant and universal nature of both quantities, we first note that though c^2 has dimensions of squared velocity it represents a 4-dimensional Lorentz scalar rather than the squared of a privileged 3-velocity. Similarly, we may see l_P^2 as a 4-d invariant with dimensions of length squared that needs not be related with any privileged 3-length. Because of dimensional compatibility with a curvature, the invariant l_P^2 could be introduced in the theory via the gravitational sector by considering departures from GR at the Planck scale motivated by quantum effects. However, the situation is not as simple as it may seem at first. In fact, an action like the one we obtained in the last section¹⁵,

$$S[g_{\mu\nu}, \psi] = \frac{\hbar}{16\pi l_P^2} \int d^4x \sqrt{-g} [R + l_P^2 R^2] + S_m[g_{\mu\nu}, \psi], \quad (1.53)$$

where $S_m[g_{\mu\nu}, \psi]$ represents the matter sector, contains the scale l_P^2 but not in the invariant form that we wished. The reason is that the field equations that follow from (1.53) are equivalent to those of the following scalar-tensor theory

$$S[g_{\mu\nu}, \varphi, \psi] = \frac{\hbar}{16\pi l_P^2} \int d^4x \sqrt{-g} \left[(1 + \varphi)R - \frac{1}{4l_P^2} \varphi^2 \right] + S_m[g_{\mu\nu}, \psi], \quad (1.54)$$

which given the identification $\phi = 1 + \varphi$ coincides with the case $w = 0$ of Brans-Dicke theory with a non-zero potential $V(\phi) = \frac{1}{4l_P^2}(\phi - 1)^2$. As is well-known and was explicitly shown in Section 1.3, in Brans-Dicke theory the observed Newton's constant is promoted to the status of field, $G_{eff} \sim G/\phi$. The scalar field allows the effective Newton's constant G_{eff} to dynamically change in time and in space. As a result the corresponding effective Planck length, $\tilde{l}_P^2 = l_P^2/\phi$, would also vary in space and time. This is quite different from the assumed constancy

¹⁵Restoring missing factors of c in (1.24), we find that $\frac{1}{16\pi G} = \frac{\hbar}{16\pi l_P^2}$ and, therefore, $\kappa^2 = 8\pi l_P^2/\hbar$.

and universality of the speed of light in special relativity, which is implicit in our construction of the total action. In fact, our action has been constructed assuming the Einstein equivalence principle (EEP), whose validity guarantees that the observed speed of light is a true constant and universal invariant, not a field¹⁶ like in varying speed of light theories [Magueijo, 2003] (recall also that Nordström’s first scalar theory was motivated by the constancy of the speed of light). The situation does not improve if we introduce higher curvature invariants in (1.54). We thus see that the introduction of the Planck length in the gravitational sector in the form of a universal constant like the speed of light is not a trivial issue. The introduction of curvature invariants suppressed by powers of $R_P = 1/l_P^2$ unavoidably generates new degrees of freedom which turn Newton’s constant into a dynamical field.

Is it then possible to modify the gravity Lagrangian adding Planck-scale corrected terms without turning Newton’s constant into a dynamical field? The answer to this question is in the affirmative. One must first note that metricity and affinity are a priori logically independent concepts [Zanelli, 2005]. If we construct the theory *à la Palatini*, that is in terms of a connection not a priori constrained to be given by the Christoffel symbols, then the resulting equations do not necessarily contain new dynamical degrees of freedom (as compared to GR), and the Planck length may remain space-time independent in much the same way as the speed of light and the coupling constants of the standard model, as required by the EEP. A natural alternative, therefore, seems to be to consider (1.53) in the Palatini formulation. The field equations that follow from (1.53) when metric and connection are varied independently are [Olmo, 2011a]

$$f_R R_{\mu\nu}(\Gamma) - \frac{1}{2} f g_{\mu\nu} = \kappa^2 T_{\mu\nu} \quad (1.55)$$

$$\nabla_\alpha (\sqrt{-g} f_R g^{\beta\gamma}) = 0, \quad (1.56)$$

where $f = R + R^2/R_P$, $f_R \equiv \partial_R f = 1 + 2R/R_P$, $R_P = 1/l_P^2$, and $\kappa^2 = 8\pi l_P^2/\hbar$. The connection equation (1.56) can be easily solved after noticing that the trace of (1.55) with $g^{\mu\nu}$,

$$R f_R - 2f = \kappa^2 T, \quad (1.57)$$

represents an algebraic relation between $R \equiv g^{\mu\nu} R_{\mu\nu}(\Gamma)$ and T , which generically implies that $R = R(T)$ and hence $f_R = f_R(R(T))$ [from now on we denote $f_R(T) \equiv f_R(R(T))$]. For the particular Lagrangian (1.53), we find that $R = -\kappa^2 T$, like in GR. This relation implies that (1.56) is just a first order equation for the connection that involves the matter, via the trace T , and the metric. The connection turns out to be the Levi-Civita connection of an auxiliary metric,

$$\Gamma_{\mu\nu}^\alpha = \frac{h^{\alpha\beta}}{2} (\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}), \quad (1.58)$$

which is conformally related with the physical metric, $h_{\mu\nu} = f_R(T) g_{\mu\nu}$. Now that the connection has been expressed in terms of $h_{\mu\nu}$, we can rewrite (1.55)

¹⁶If the Einstein equivalence principle is true, then all the coupling constants of the standard model are constants, not fields [Will, 2005].

as follows

$$G_{\mu\nu}(h) = \frac{\kappa^2}{f_R(T)} T_{\mu\nu} - \Lambda(T) h_{\mu\nu} \quad (1.59)$$

where $\Lambda(T) \equiv (Rf_R - f)/(2f_R^2) = (\kappa^2 T)^2/R_P$, and looks like Einstein's theory for the metric $h_{\mu\nu}$ with a slightly modified source. This set of equations can also be written in terms of the physical metric $g_{\mu\nu}$ as follows

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) &= \frac{\kappa^2}{f_R} T_{\mu\nu} - \frac{Rf_R - f}{2f_R} g_{\mu\nu} - \frac{3}{2(f_R)^2} \left[\partial_\mu f_R \partial_\nu f_R - \frac{1}{2}g_{\mu\nu}(\partial f_R)^2 \right] + \\ &\quad \frac{1}{f_R} [\nabla_\mu \nabla_\nu f_R - g_{\mu\nu} \square f_R] . \end{aligned} \quad (1.60)$$

In this last representation, one can use the notation introduced in (1.29) and (1.30) to show that these field equations coincide with those of a Brans-Dicke theory with parameter $w = -3/2$ (see eq.(1.22)). Note that all the functions $f(R)$, R , and f_R that appear on the right hand side of (1.60) are functions of the trace T . This means that the modified dynamics of (1.60) is due to the new matter terms induced by the trace T of the matter, not to the presence of new dynamical degrees of freedom. This also guarantees that, unlike for the $w \neq -3/2$ Brans-Dicke theories, for the $w = -3/2$ theory Newton's constant is indeed a constant.

From the structure of the field equations (1.59) and (1.60), and the relation $g_{\mu\nu} = (1/f_R)h_{\mu\nu}$, it follows that $g_{\mu\nu}$ is affected by the matter-energy in two different ways. The first contribution corresponds to the cumulative effects of matter, and the second contribution is due to the dependence on the local density distributions of energy and momentum. This can be seen by noticing that the structure of the equations (1.59) that determine $h_{\mu\nu}$ is similar to that of GR, which implies that $h_{\mu\nu}$ is determined by integrating over all the sources (gravity as a cumulative effect). Besides that, $g_{\mu\nu}$ is also affected by the local sources through the factor $f_R(T)$. To illustrate this point, consider a region of the spacetime containing a total mass M and filled with sources of low energy-density as compared to the Planck scale ($|\kappa^2 T/R_P| \ll 1$). For the quadratic model $f(R) = R + R^2/R_P$, in this region (1.59) boils down to $G_{\mu\nu}(h) = \kappa^2 T_{\mu\nu} + O(\kappa^2 T/R_P)$, and $h_{\mu\nu} \approx (1 + O(\kappa^2 T/R_P))g_{\mu\nu}$, which implies that the GR solution is a very good approximation. This confirms that $h_{\mu\nu}$ is determined by an integration over the sources, like in GR. Now, if this region is traversed by a particle of mass $m \ll M$ but with a non-negligible ratio $\kappa^2 T/R_P$, then the contribution of this particle to $h_{\mu\nu}$ can be neglected, but its effect on $g_{\mu\nu}$ via the factor $f_R = 1 - \kappa^2 T/R_P$ on the region that supports the particle (its classical trajectory) is important. This phenomenon is analogous to that described in the so-called Rainbow Gravity [Magueijo and Smolin, 2004], an approach to the phenomenology of quantum gravity based on a non-linear implementation of the Lorentz group to allow for the coexistence of a constant speed of light and a maximum energy scale (the flat space version of that theory is known as Doubly Special Relativity [Amelino-Camelia, 2002, Magueijo and Smolin, 2002, Magueijo and Smolin, 2003, Amelino-Camelia and Smolin, 2009]). In Rainbow

Gravity, particles of different energies (energy-densities in our case) perceive different metrics.

1.4.2 The early-time cosmology of Palatini $f(R)$ models.

The quadratic Palatini model introduced above turns out to be virtually indistinguishable from GR at energy densities well below the Planck scale. It is thus natural to ask if this theory presents any particularly interesting feature at Planck scale densities. A natural context where this question can be explored is found in the very early universe, when the matter energy-density tends to infinity as we approach the big bang.

In a spatially flat, homogeneous, and isotropic universe, with line element $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$, filled with a perfect fluid with constant equation of state $P = w\rho$ and density ρ , the Hubble function that follows from (1.60) (or (1.59)) takes the form

$$H^2 = \frac{1}{6f_R} \frac{[f + (1+3w)\kappa^2\rho]}{[1 + \frac{3}{2}\Delta]^2}, \quad (1.61)$$

where $H = \dot{a}/a$, and $\Delta = -(1+w)\rho\partial_\rho f_R/f_R = (1+w)(1-3w)\kappa^2\rho f_{RR}/(f_R(Rf_{RR} - f_R))$. In GR, (1.61) boils down to $H^2 = \kappa^2\rho/3$. Since the matter conservation equation for constant w leads to $\rho = \rho_0/a^{3(1+w)}$, in GR we find that $a(t) = a_0 t^{\frac{2}{3(1+w)}}$, where ρ_0 and a_0 are constants. This result indicates that if the universe is dominated by a matter source with $w > -1$, then at $t = 0$ the universe has zero physical volume, the density is infinite, and all curvature scalars blow up, which indicates the existence of a big bang singularity. The quadratic Palatini model introduced above, however, can avoid this situation. For that model, (1.61) becomes [Barragan et al., 2009a, Barragan et al., 2009b, Olmo, 2010]

$$H^2 = \frac{\kappa^3\rho}{3} \frac{\left(1 + \frac{2R}{R_P}\right) \left(1 + \frac{1-3w}{2} \frac{R}{R_P}\right)}{\left[1 - (1+3w)\frac{R}{R_P}\right]^2}. \quad (1.62)$$

This expression recovers the linear dependence on ρ of GR in the limit $|R/R_P| \ll 1$. However, if R reaches the value $R_b = -R_P/2$, then H^2 vanishes and the expansion factor $a(t)$ reaches a minimum. This occurs for $w > 1/3$ if $R_P > 0$ and for $w < 1/3$ if $R_P < 0$. The existence of a non-zero minimum for the expansion factor implies that the big bang singularity is avoided. The avoidance of the big bang singularity indicates that the time coordinate can be extended backwards in time beyond the instant $t = 0$. This means that in the past the universe was in a contracting phase which reached a minimum and bounced off to the expanding phase that we find in GR.

We mentioned at the beginning of this section that the avoidance of the big bang singularity is a basic requirement for any acceptable quantum theory of gravity. Our procedure to construct a quantum-corrected theory of gravity in which the Planck length were a universal invariant similar to the speed

of light has led us to a cosmological model which replaces the big bang by a cosmic bounce. To obtain this result, it has been necessary to resort to the Palatini formulation of the theory. In this sense, it is important to note that the metric formulation of the quadratic curvature model discussed here, besides turning the Planck length into a dynamical field, is unable to avoid the big bang singularity. In fact, in metric formalism, all quadratic models of the form $R + (aR^2 + bR_{\mu\nu}R^{\mu\nu})/R_P$ that at late times tend to a standard Friedmann-Robertson-Walker cosmology begin with a big bang singularity. Palatini theories, therefore, appear as a potentially interesting framework to discuss quantum gravity phenomenology.

1.4.3 A Palatini action for loop quantum cosmology

Growing interest in the dynamics of the early-universe in Palatini theories has arisen, in part, from the observation that the effective equations of loop quantum cosmology (LQC) [Bojowald, 2005, Ashtekar et al., 2006a, Ashtekar et al., 2006b, Ashtekar et al., 2006c, Ashtekar, 2007, Szulc et al., 2007], a Hamiltonian approach to quantum gravity based on the non-perturbative quantization techniques of loop quantum gravity [Rovelli, 2004, Thiemann, 2007], could be exactly reproduced by a Palatini $f(R)$ Lagrangian [Olmo and Singh, 2009]. In LQC, non-perturbative quantum gravity effects lead to the resolution of the big bang singularity by a quantum bounce without introducing any new degrees of freedom. Though fundamentally discrete, the theory admits a continuum description in terms of an effective Hamiltonian that in the case of a homogeneous and isotropic universe filled with a massless scalar field leads to the following modified Friedmann equation

$$3H^2 = 8\pi G\rho \left(1 - \frac{\rho}{\rho_{crit}}\right), \quad (1.63)$$

where $\rho_{crit} \approx 0.41\rho_{Planck}$. At low densities, $\rho/\rho_{crit} \ll 1$, the background dynamics is the same as in GR, whereas at densities of order ρ_{crit} the non-linear new matter contribution forces the vanishing of H^2 and hence a cosmic bounce. This singularity avoidance seems to be a generic feature of loop-quantized universes [Singh, 2009].

Palatini $f(R)$ theories share with LQC an interesting property: the modified dynamics that they generate is not the result of the existence of new dynamical degrees of freedom but rather it manifests itself by means of non-linear contributions produced by the matter sources, which contrasts with other approaches to quantum gravity and to modified gravity. This similarity makes it tempting to put into correspondence Eq.(1.63) with the corresponding $f(R)$ equation (1.60). Taking into account the trace equation (1.57), which for a massless scalar becomes $Rf_R - 2f = 2\kappa^2\rho$ and implies that $\rho = \rho(R)$, one finds that a Palatini $f(R)$ theory able to reproduce the LQC dynamics (1.63) must satisfy the differential equation

$$f_{RR} = -f_R \left(\frac{Af_R - B}{2(Rf_R - 3f)A + RB} \right) \quad (1.64)$$

where $A = \sqrt{2(Rf_R - 2f)(2R_c - [Rf_R - 2f])}$, $B = 2\sqrt{R_c f_R(2Rf_R - 3f)}$, and $R_c \equiv \kappa^2 \rho_c$. If one imposes the boundary condition $\lim_{R \rightarrow 0} f_R \rightarrow 1$ at low curvatures, and $\ddot{a}_{LQC} = \ddot{a}_{Pal}$ (where \ddot{a} represents the acceleration of the expansion factor) at $\rho = \rho_c$, the solution to this equation is unique. The solution was found numerically [Olmo and Singh, 2009], though the following function can be regarded as a very accurate approximation to the LQC dynamics from the GR regime to the non-perturbative bouncing region (see Fig.1.1)

$$\frac{df}{dR} = -\tanh\left(\frac{5}{103} \ln\left[\left(\frac{R}{12R_c}\right)^2\right]\right) \quad (1.65)$$

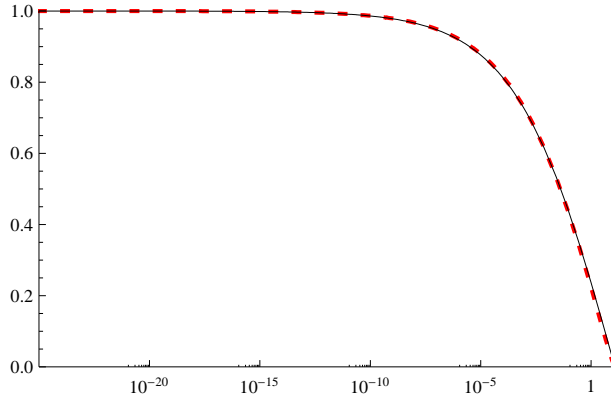


Figure 1.1: Vertical axis: df/dR ; Horizontal axis: R/R_c . Comparison of the numerical solution with the interpolating function (1.65). The dashed line represents the numerical curve.

This result is particularly important because it establishes a direct link between the Palatini approach to modified gravity and a cosmological model derived from non-perturbative quantization techniques.

1.4.4 Beyond Palatini $f(R)$ theories.

Nordström's second theory was a very interesting theoretical exercise that successfully allowed to implement the Einstein equivalence principle in a relativistic scalar theory. However, among other limitations, that theory did not predict any new gravitational effect for the electromagnetic field. In a sense, Palatini $f(R)$ theories suffer from this same limitation. Since their modified dynamics is due to new matter contributions that depend on the trace of the stress-energy tensor, for traceless fields such as a radiation fluid or the electromagnetic field, the theory does not predict any new effect. This drawback can be avoided if one adds to the Palatini Lagrangian a new piece dependent on the squared Ricci tensor, $R_{\mu\nu}R^{\mu\nu}$, where we assume $R_{\mu\nu} = R_{\nu\mu}$

[Olmo et al., 2009, Barragan and Olmo, 2010]. In particular, the following action

$$S[g_{\mu\nu}, \Gamma_{\alpha\beta}^{\mu}, \psi] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + a \frac{R^2}{R_P} + \frac{R_{\mu\nu} R^{\mu\nu}}{R_P} \right] + S_m[g_{\mu\nu}, \psi], \quad (1.66)$$

implies that $R = R(T)$ but $Q \equiv R_{\mu\nu} R^{\mu\nu} = Q(T_{\mu\nu})$, i.e., the scalar Q has a more complicated dependence on the stress-energy tensor of matter than the trace. For instance, for a perfect fluid, one finds

$$\frac{Q}{2R_P} = - \left(\kappa^2 P + \frac{\tilde{f}}{2} + \frac{R_P}{8} \tilde{f}_R^2 \right) + \frac{R_P}{32} \left[3 \left(\frac{R}{R_P} + \tilde{f}_R \right) - \sqrt{\left(\frac{R}{R_P} + \tilde{f}_R \right)^2 - \frac{4\kappa^2(\rho + P)}{R_P}} \right]^2, \quad (1.67)$$

where $\tilde{f} = R + aR^2/R_P$ and R is a solution of $R\tilde{f}_R - 2\tilde{f} = \kappa^2 T$. From this it follows that even if one deals with a radiation fluid ($P = \rho/3$) or with a traceless field, the Palatini action (1.66) generates modified gravity without introducing new degrees of freedom.

For this model, it has been shown that completely regular bouncing solutions exist for both isotropic and anisotropic homogeneous cosmologies filled with a perfect fluid. In particular, one finds that for $a < 0$ the interval $0 \leq w \leq 1/3$ is always included in the family of bouncing solutions, which contains the dust and radiation cases. For $a \geq 0$, the fluids yielding a non-singular evolution are restricted to $w > \frac{a}{2+3a}$, which implies that the radiation case $w = 1/3$ is always nonsingular. For a detailed discussion and classification of the non-singular solutions depending on the value of the parameter a and the equation of state w , see [Barragan and Olmo, 2010].

As an illustration, consider a universe filled with radiation, for which $R = 0$. In this case, the function Q boils down to [Barragan and Olmo, 2010]

$$Q = \frac{3R_P^2}{8} \left[1 - \frac{8\kappa^2\rho}{3R_P} - \sqrt{1 - \frac{16\kappa^2\rho}{3R_P}} \right]. \quad (1.68)$$

This expression recovers the GR value at low curvatures, $Q \approx 4(\kappa^2\rho)^2/3 + 32(\kappa^2\rho)^3/9R_P + \dots$ but reaches a maximum $Q_{max} = 3R_P^2/16$ at $\kappa^2\rho_{max} = 3R_P/16$, where the squared root of (1.68) vanishes. It can be shown that at ρ_{max} the shear also takes its maximum, namely, $\sigma_{max}^2 = \sqrt{3/16}R_P^{3/2}(C_{12}^2 + C_{23}^2 + C_{31}^2)$, which is always finite, and the expansion vanishes producing a cosmic bounce regardless of the amount of anisotropy (see Fig.1.2). The model (1.66), therefore, avoids the well-known problems of anisotropic universes in GR, where anisotropies grow faster than the energy density during the contraction phase leading to a singularity that can only be avoided by sources with $w > 1$.

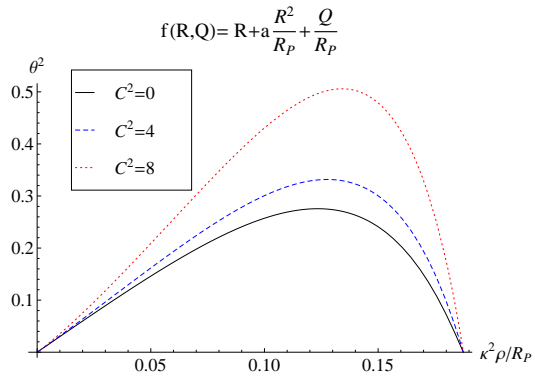


Figure 1.2: Evolution of the expansion as a function of $\kappa^2 \rho / R_P$ in radiation universes with low anisotropy, which is controlled by the combination $C^2 = C_{12}^2 + C_{23}^2 + C_{31}^2$. The case with $C^2 = 0$ corresponds to the isotropic flat case, $\theta^2 = 9H^2$.

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